Twist decomposition of nonlocal light-ray operators and harmonic tensor functions[1]

B. Geyer and M. Lazar

Naturwissenschaftlich-Theoretisches Zentrum,
Universität Leipzig, Augustusplatz 10, D–04109 Leipzig, Germany

1 Introduction

Light–cone(LC) dominated hard scattering processes, like deep inelastic scattering and deeply virtual Compton scattering, and various hadronic wave functions, e.g., vector meson amplitudes, appearing in (semi) exclusive processes are most effectively described by the help of the *nonlocal* light–cone expansion [1].Thereby, the *same* nonlocal LC–operator, and its anomalous dimension, is related to *different* phenomenological distribution amplitudes, and their Q^2–evolution kernels [2]. Growing experimental precision requires the consideration of higher twist contributions. Therefore, it is necessary to decompose the nonlocal LC–operators according to their twist content.

Unfortunately, 'geometric' twist (τ) = dimension (d)− spin (j), introduced for the *local* LC–operators [3] cannot be extended directly to nonlocal LC–operators. Therefore, motivated by LC–quantization and by kinematic phenomenology the notion of 'dynamic' twist (t) was introduced counting powers Q^{2-t} of the momentum transfer [4]. However, this notion is defined only for *matrix elements* of operators, is *not* Lorentz invariant and its relation to 'geometric' twist is complicated, cf. [5].

Here, we carry on a systematic procedure to uniquely decompose nonlocal LC–operators into harmonic operators of well defined geometric twist, cf. Ref. [6]. This will be demonstrated for tensor operators of low rank, namely (pseudo)scalars, (axial) vectors and (skew)symmetric tensors Thereby, various harmonic tensor operators (in $D = 2h$ space-time dimensions) are introduced being related to specific infinite series of symmetry classes which are defined by corresponding Young tableaus. In the scalar case these LC–operators are series of the well-known harmonic polynomials corresponding to (symmetric) tensor representations of the orthogonal group $SO(D)$, cf. [7]. Symmetric tensor operators of rank 2 are considered as an example.

[1]Contribution to Proceedings of International Symposium "Quantum Theory and Symmetries", Goslar, 18. – 22. July 1999

2 General procedure

Let us generically denote by $O_\Gamma(\kappa_1\tilde{x},\kappa_2\tilde{x})$ with $\tilde{x}^2 = 0$ any of the following bilocal light-ray operators, either the quark operators $:\overline{\psi}(\kappa_1\tilde{x})\Gamma U(\kappa_1\tilde{x},\kappa_2\tilde{x})\psi(\kappa_2\tilde{x}):$ with $\Gamma = \{1,\gamma_\alpha,\sigma_{\alpha\beta},\gamma_5\gamma_\alpha,\gamma_5\}$ or the gluon operators $:F_\mu{}^\rho(\kappa_1\tilde{x})U(\kappa_1\tilde{x},\kappa_2\tilde{x})F_{\nu\rho}(\kappa_2\tilde{x}):$ and $:F_\mu{}^\rho(\kappa_1\tilde{x})U(\kappa_1\tilde{x},\kappa_2\tilde{x})\widetilde{F}_{\nu\rho}(\kappa_2\tilde{x}):$, where $\psi(x), F_{\mu\nu}(x)$ and $\widetilde{F}_{\mu\nu}(x)$ are the quark field, the gluon field strenght and its dual, respectively, and the path ordered phase factor is given by $U(\kappa_1 x,\kappa_2 x) = \mathcal{P}\exp\{-\mathrm{i}g\int_{\kappa_2}^{\kappa_1}\mathrm{d}\kappa\, x^\mu A_\mu(\kappa x)\}$ with the gauge potential $A_\mu(x)$. The general procedure of decomposing these operators into operators of definite twist consists of the following steps:

(1) *Expansion* of the nonlocal operators for *arbitrary* values of x into a Taylor series of *local* tensor operators having definite rank n and canonical dimension d:

$$O_\Gamma(\kappa_1 x,\kappa_2 x) = \textstyle\sum_{n=0}^{\infty}(n!)^{-1}x^{\mu_1}\ldots x^{\mu_n}O_{\Gamma\,\mu_1\ldots\mu_n}(\kappa_1,\kappa_2); \tag{1}$$

the local operators are defined by (the brackets $(\ldots)$ denote total symmetrization)

$$O_{\Gamma\,\mu_1\ldots\mu_n}(\kappa_1,\kappa_2) \equiv \big[\Phi(x)\Gamma\mathsf{D}_{(\mu_1}\ldots\mathsf{D}_{\mu_n)}\Phi(x)\big]\big|_{x=0}, \tag{2}$$

with generalized covariant derivatives $\mathsf{D}_\mu(\kappa_1,\kappa_2) \equiv \kappa_1(\overleftarrow{\partial_\mu} - \mathrm{i}gA_\mu) + \kappa_2(\overrightarrow{\partial_\mu} + \mathrm{i}gA_\mu)$.

(2) *Decomposition* of the local operators (2) into tensors being *irreducible* under the (orthochronous) Lorentz group, i.e. having twist $\tau = d - j$. These are *traceless* tensors being classified according to their *symmetry class*. The latter is determined by a (normalized) *Young-Operator* $\mathcal{Y}_{[m]} = f_{[m]}\mathcal{QP}/n!$, where $[m] = (m_1 \geq m_2 \geq \ldots \geq m_r)$ with $\sum_{i=1}^r m_i = n$ defines a Young pattern, and $\mathcal{P}$ and $\mathcal{Q}$ denote symmetrizations and antisymmetrizations w.r.to the horizontal and vertical permutations of a standard tableau, respectively; $f_{[m]}$ is the number of standard tableau's to $[m]$.

This decomposition can be done for any dimension $D = 2h$ of the complex orthogonal group $SO(2h, C)$. Then, the allowed Young patterns are restricted by $\ell_1 + \ell_2 \leq 2h$ (ℓ_i: lenght of columns of $[m]$). Since the operators (2) are totally symmetric w.r.t. μ_i's only the following symmetry types, depending on the additional tensor structure Γ, are of relevance (lower spins j are related to trace terms):

(i)	$[m] = (n)$	$j = n, n-2, n-4, \ldots,$	$f_{(n)} = 1,$
(ii)	$[m] = (n,1)$	$j = n, n-1, n-2, \ldots,$	$f_{(n,1)} = n,$
(iii)	$[m] = (n,1,1)$	$j = n, n-1, n-2, \ldots,$	$f_{(n,1,1)} = n(n+1)/2,$
(iv)	$[m] = (n,2)$	$j = n, n-1, n-2, \ldots,$	$f_{(n,2)} = (n+1)(n+2)/2.$

If multiplied by $x^{\mu_1}\ldots x^{\mu_n}$ according to Eq. (1) these representations may be characterized by harmonic tensor polynomials of order n, cf. Sect. 3 below.

(3) *Resummation* of the infinite series (for any n) of irreducible tensor harmonics of equal twist τ and symmetry type $\mathcal{Y}_{[\mathrm{m}]}$ to nonlocal *harmonic operators of definite twist*. Thereby, the phase factors may be reconstructed on the expense of (two) additional integrations over parameters multiplying the κ–variables. As a result one obtains a decomposition of the original nonlinear operator into a series of operators with growing twist, each being defined by subtracting appropriate traces, cf. Sect. 4 below.

(4) *Projection* onto the light–cone, $x \to \tilde{x} = x + \eta(x\eta)\big(\sqrt{1 - x^2/(x\eta)^2} - 1\big), \quad \eta^2 = 1$, leads to the required twist decomposition:

$$O_\Gamma(\kappa_1\tilde{x}, \kappa_2\tilde{x}) \;=\; \textstyle\sum_{\tau_i=\tau_{\rm min}}^{\tau_{\rm max}} O_\Gamma^{\tau_i}(\kappa_1\tilde{x}, \kappa_2\tilde{x}). \tag{3}$$

This sum terminates since most of the traces are proportional to x^2 and, therefore, vanish on the light–cone.
Let us remark that step (3) and (4) may be interchanged without changing the result. In fact, this corresponds to another formalism acting on the complex light–cone [7] being used for the construction of conformal tensors by Dobrev and Ganchev [8].

3 Harmonic tensor polynomials

The construction of irreducible tensors may start with traceless tensors $\stackrel{\circ}{T}_{\Gamma(\mu_1 \ldots \mu_n)}$, with Γ indicating additional tensor indices, which afterwards have to be subjected to the symmetry requirements determined by the symmetry classes (i) – (iv). Let us start with a generic tensor, not being traceless, whose symmetrized indices are contracted by x^{μ_i}'s: $T_{\Gamma n}(x) = x^{\mu_1} \ldots x^{\mu_n} T_{\Gamma(\mu_1 \ldots \mu_n)}$. The conditions for scalar, (axial) vector resp. (skew)symmetric tensors to be traceless read:

$$\Box \stackrel{\circ}{T}_{\Gamma\, n}(x) \;=\; 0, \tag{4}$$

$$\partial^\alpha \stackrel{\circ}{T}_{\alpha\, n}(x) = 0 \quad \text{resp.} \quad \partial^\alpha \stackrel{\circ}{T}_{\alpha\beta\, n}(x) = \; 0 \; = \partial^\beta \stackrel{\circ}{T}_{\alpha\beta\, n}(x) \quad \text{and} \quad g^{\alpha\beta} \stackrel{\circ}{T}_{\alpha\beta\, n}(x) = 0. \tag{5}$$

The general solutions of these equations in $D = 2h$ dimensions are:
(1) *Scalar harmonic polynomials* (cf. [9]):

$$\stackrel{\circ}{T}_n(x) = \sum_{k=0}^{[\frac{n}{2}]} \frac{(h{+}n{-}k{-}2)!}{k!(h{+}n{-}2)!} \left(\frac{-x^2}{4}\right)^k \Box^k T_n(x) \equiv H_n^{(2h)}\big(x^2|\Box\big)\, T_n(x) \tag{6}$$

(2) *Vector harmonic polynomials*:

$$\stackrel{\circ}{T}_{\alpha n}(x) = \Big\{\delta_\alpha^\beta - \frac{1}{(h{+}n{-}1)(2h{+}n{-}3)} \Big((h-2+x\partial)x_\alpha\partial^\beta - \tfrac{1}{2}x^2\partial_\alpha\partial^\beta\Big)\Big\} H_n^{(2h)}\big(x^2|\Box\big) T_{\beta n}(x) \tag{7}$$

(3) *Skew tensor harmonic polynomials*:

$$\begin{aligned}\stackrel{\circ}{T}_{[\alpha\beta]n}(x) = \Big\{&\delta^\mu_{[\alpha}\delta^\nu_{\beta]} + \frac{2}{(h{+}n{-}1)(2h{+}n{-}4)}\Big((h-2+x\partial)x_{[\alpha}\delta^{[\mu}_{\beta]}\partial^{\nu]} - \tfrac{1}{2}x^2\partial_{[\alpha}\delta^{[\mu}_{\beta]}\partial^{\nu]}\Big) \\ &- \frac{2}{(h{+}n{-}1)(2h{+}n{-}4)(2h{+}n{-}2)} x_{[\alpha}\partial_{\beta]}x^{[\mu}\partial^{\nu]}\Big\} H_n^{(2h)}(x^2|\Box)\, T_{[\mu\nu]n}(x)\end{aligned} \tag{8}$$

(4) *Symmetric tensor harmonic polynomials*:

$$\begin{aligned}\stackrel{\circ}{T}_{(\alpha\beta)n}(x) = \Big\{&\delta^\mu_{(\alpha}\delta^\nu_{\beta)} + a_n g_{\alpha\beta}x^{(\mu}\partial^{\nu)} - 2g_n x_{(\alpha}\delta^{(\mu}_{\beta)}\partial^{\nu)} - 2b_n x_{(\alpha}x^{(\mu}\partial_{\beta)}\partial^{\nu)} + c_n x^2 \delta^{(\mu}_{(\alpha}\partial_{\beta)}\partial^{\nu)} \\ &- d_n x^2 x_{(\alpha}\partial_{\beta)}\partial^\mu\partial^\nu + e_n x_\alpha x_\beta \partial^\mu\partial^\nu + f_n x^2 x^{(\mu}\partial^{\nu)}\partial_\alpha\partial_\beta - k_n x^2 g_{\alpha\beta}\partial^\mu\partial^\nu \\ &+ h_n (x^4/4)\partial_\alpha\partial_\beta\partial^\mu\partial^\nu\Big\}\Big(\delta^\rho_\mu\delta^\sigma_\nu - (2h)^{-1}g_{\mu\nu}g^{\rho\sigma}\Big) H_n^{(2h)}(x^2|\Box)\, T_{(\rho\sigma)n}(x)\end{aligned} \tag{9}$$

where, up to the common factor $[(h-1)(h+n)(h+n-1)(2h+n-2)(2h+n-3)]^{-1}$, the coefficients are given by

$a_n = (h+n-1)(h+n-2)(2h+n-3)$, $\quad b_n = (h+n-2)(2h+n-3)$,
$c_n = [h(h+n-1)-n+2](2h+n-3)$, $\quad d_n = [h(h+n-4)-2(n-3)]$,
$e_n = [h(h+n-2)-n+3](h+n-2)$, $\quad f_n = (2h+n-3)$,
$g_n = [h(h+n-1)-n+1](h+n-2)(2h+n-3)$, $\quad h_n = (h-3)$
$k_n = [h(h+n-3)-2n+3]+(h+n)(h+n-1)/2$.

These coefficients look quite complicated. However, in $D=4$ dimensions they simplify considerably. The extension to tensors of higher rank is obvious, but cumbersome. Up to now no simple building principle has been found, and a general theory, to the best of our knowledge, does not exist.

4 Twist decomposition: An example

Besides the scalar case these harmonic tensor polynomials are not irreducible with respect to $SO(2h,C)$. Irreducible polynomials are obtained by (anti)symmetrization according to the Young patterns (i) – (iv); this may be achieved by acting on it with corresponding differential operators, multiplied by the weights $f_{[m]}$. Afterwards, in order to get the (local) operators on the light–cone, one takes the limit $x \to \tilde{x}$. The nonlocal LC–operators are obtained by resummation according to Eq. (1).

For the quark operators that procedure has been already considered in [6]. Here we shall consider the gluon operators being tensors of rank 2. The relevant Young patterns are given by (i) – (iv). Of course, in physical applications they appear sometimes also multiplied with x_α or/and x_β; then higher patterns are irrelevant.

Let us consider totally symmetric tensors leading to much simpler symmetric harmonic tensor polynomials then those given by the general expression (9):

$$T^{(\mathrm{i})}_{(\alpha\beta)n}(x) = \frac{1}{(n+2)(n+1)}\partial_\alpha\partial_\beta \overset{\circ}{T}_{n+2}(x) = \frac{1}{(n+2)(n+1)}\sum_{k=0} \frac{(-1)^k(h+n-k)!}{4^k k!(h+n)!}\Big\{x^{2k}\partial_\alpha\partial_\beta + 2kx^{2(k-1)}\big(g_{\alpha\beta}+2x_{(\alpha}\partial_{\beta)}\big) + 4k(k-1)x_\alpha x_\beta x^{2(k-2)}\Big\}\Box^k T_{n+2}(x). \tag{10}$$

Related to the number of partial derivatives this sum contains symmetric tensor, vector and scalar polynomials of lower degree. Of course, if projected onto the light-cone only those terms survive which do not depend on x^2. They are given by

$$T^{(\mathrm{i})}_{(\alpha\beta)n}(\tilde{x}) = \frac{1}{(n+2)(n+1)(h+n)(h+n-1)}\mathrm{d}_\alpha\mathrm{d}_\beta T_{n+2}(\tilde{x}), \tag{11}$$

with the 'interior' derivative $\mathrm{d}_\mu = [(h-1+x\partial)\partial_\mu - \frac{1}{2}x_\mu\Box]\big|_{x=\tilde{x}}$ on the light-cone (cf. Ref. [7]); because of $\mathrm{d}^2=0$ the conditions of tracelessness on the light-cone, $\mathrm{d}^\alpha T^{(\mathrm{i})}_{(\alpha\beta)n}(\tilde{x}) = 0 = \mathrm{d}^\beta T^{(\mathrm{i})}_{(\alpha\beta)n}(\tilde{x})$ and $g^{\alpha\beta}T^{(\mathrm{i})}_{(\alpha\beta)n}(\tilde{x}) = 0$, are trivially fulfilled.

These polynomials have well defined twist $\tau=2$. They are obtained from the original operator $T_{\alpha\beta}(x) = \frac{1}{(n+2)(n+1)}\partial_\alpha\partial_\beta T_{n+2}(x)$ by subtracting the traces being of twist 4 and 6. These traces, corresponding to irreducible representations, are

$$T^{\rm tw4(i)a}_{(\alpha\beta)n}(\tilde x) = \frac{1}{2(n+2)(n+1)(h+n)} g_{\alpha\beta}\Box T_{n+2}(x)\Big|_{x=\tilde x}, \qquad (12)$$

$$T^{\rm tw4(i)b}_{(\alpha\beta)n}(\tilde x) = \frac{1}{(n+2)(n+1)(h+n)} \Big(x_{(\alpha}\partial_{\beta)}\Box - \frac{1}{2(h+n-2)} x_\alpha x_\beta \Box^2 \Big) T_{n+2}(x)\Big|_{x=\tilde x}, \qquad (13)$$

$$T^{\rm tw6(i)a}_{(\alpha\beta)n}(\tilde x) = -\frac{1}{4(n+2)(n+1)(h+n)(h+n-1)} x_\alpha x_\beta \Box^2 T_{n+2}(x)\Big|_{x=\tilde x}, \qquad (14)$$

$$T^{\rm tw6(i)b}_{(\alpha\beta)n}(\tilde x) = \frac{1}{2(n+2)(n+1)(h+n)(h+n-2)} x_\alpha x_\beta \Box^2 T_{n+2}(x)\Big|_{x=\tilde x}. \qquad (15)$$

For the gluon operator $G_{(\alpha\beta)}(\kappa_1\tilde x, \kappa_2\tilde x) = F^{\ \rho}_{(\alpha}(\kappa_1\tilde x) U(\kappa_1\tilde x, \kappa_2\tilde x) F_{\beta)\rho}(\kappa_2\tilde x)$ the expressions (11) may be resummed leading, if restricted to $D = 4$, to the following one-parameter integral representation ($G(\kappa_1\tilde x, \kappa_2\tilde x) \equiv x^\alpha x^\beta G_{(\alpha\beta)}(\kappa_1\tilde x, \kappa_2\tilde x)$):

$$\begin{aligned} G^{\rm tw2(i)}_{(\alpha\beta)}(\kappa_1\tilde x, \kappa_2\tilde x) = \int_0^1 {\rm d}\lambda \Big\{ &(1-\lambda)\partial_\alpha\partial_\beta - (1-\lambda+\lambda\ln\lambda)\Big(\tfrac{1}{2} g_{\alpha\beta} + x_{(\alpha}\partial_{\beta)}\Big)\Box \\ &- \tfrac{1}{4}\big(2(1-\lambda) + (1+\lambda)\ln\lambda\big) x_\alpha x_\beta \Box^2 \Big\} G(\kappa_1\lambda x, \kappa_2\lambda x)\Big|_{x=\tilde x}. \end{aligned} \qquad (16)$$

Analoguous expressions result for (12) – (15). This completes the twist decomposition of $G^{\rm (i)}_{(\alpha\beta)}(\kappa_1\tilde x, \kappa_2\tilde x)$. The decomposition of $G_{\alpha\beta}(\kappa_1\tilde x, \kappa_2\tilde x)$ w.r.to the other symmetry classes is much more complicated. A complete presentation will be given elsewhere.

The authors would like to thank D. Robaschik for many stimulating discussions.

References

[1] S.A. Anikin and O.I. Zavialov, Ann. Phys. (NY) 116 (1978) 135

[2] D. Müller, D. Robaschik, B. Geyer, F. Dittes and J. Hořejši, Fortschr. Phys. 42 (1994) 101, reprinted as hep-th/9812448

[3] D.J. Gross and S.B. Treiman, Phys. Rev. D4 (1971) 1059

[4] see, e.g., R. Jaffe and X. Ji, Nucl. Phys. B375 (1992) 527

[5] P. Ball, V. Braun, Y. Koike and K. Tanaka, Nucl. Phys. B529 (1998) 323

[6] B. Geyer, M. Lazar and D. Robaschik, Nucl. Phys. B559 (1999) 339, Nucl. Phys. B (Proc. Suppl.) 79 (1999) 560; see also J. Blümlein, B. Geyer and D. Robaschik, Nucl. Phys. B560 (1999) 283

[7] V. Bargmann and I.T. Todorov, J. Math. Phys. 18 (1977) 1141

[8] V.K. Dobrev and A.Ch. Ganchev, *Conformal operators from spinor fields: Antisymmetric tensor case*, preprint Dubna E2-82-881 (1982)

[9] see, e.g., N.Ya. Vilenkin and A.U. Klimyk, *Representations of Lie groups and Special Functions, Vol. 2*, Kluwer Academic Press, Dordrecht, 1993